

A Universal Amplitude Ratio for the $q \leq 4$ Potts Model from a Solvable Lattice Model

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The universal amplitude ratio R_ξ for the ($q \leq 4$)-state Potts model in two dimensions is determined by using results for the dilute A model in regime 1. The nature of the relationship between the Potts model and the dilute A model is discussed.

KEY WORDS: Potts model; dilute A model; universal amplitude ratio; percolation.

1. INTRODUCTION

There has recently been interest in determining universal amplitude ratios, familiar in statistical mechanics,⁽¹⁾ using the techniques and results of perturbed conformal field theory. Since an integrable perturbation corresponds to the scaling limit of a two-dimensional lattice model in statistical mechanics, these amplitudes have found direct application to the Ising model,⁽²⁾ the Potts model⁽³⁾ and the tricritical Ising model,⁽⁴⁾ for example. When the corresponding lattice model is solvable, or its universality class contains a solvable counterpart, one would hope to find some of the same amplitudes using the techniques and results of the solvable model literature. Indeed, among the integrable field theory results of ref. 2 are recovered the “thermal” amplitudes of the Ising model, known since the seventies.⁽⁵⁾ In previous papers^(6,7) universal amplitude ratios for the subleading magnetic and leading thermal perturbations of the tricritical Ising model were calculated, by considering their realization as members of

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the dilute A_L model hierarchy: the A_3 model in regime 1 and the A_4 model in regime 2, respectively. The amplitude ratios obtainable were confirmed to be identical to those found in refs. 4 and 8.

In this paper one universal amplitude ratio for the Potts model, among those given in ref. 3, is determined from the dilute A model in regime 1 by utilizing a relationship weaker than shared universality class. Preliminary results for percolation ($q = 1$) have been announced,⁽⁹⁾ and this present paper completes the study. Points of contact between the dilute A model and the $q \leq 4$ Potts model are outlined in Section 2, and expressions for the required thermodynamic quantities for the dilute A model are given. In Section 3 these connections and results are exploited to determine an expression for the amplitude ratio R_ξ^+ of the Potts model for each integer value $1 \leq q \leq 4$. For the Ising ($q = 2$) case, it is also demonstrated that this quantity can be determined from the Andrews–Baxter–Forrester⁽¹⁰⁾ model. The relationship of these results to quantum field theory results for the Potts model⁽³⁾ is discussed in Section 4.

2. THE MODELS

The dilute A_L model⁽¹¹⁾ is an L state, interaction-round-a-face model which has been solved⁽¹²⁾ in four regimes, two of which provide off-critical extensions of the unitary minimal conformal field theories. The model's adjacency diagram is that of A_L with the modification that a state may be adjacent to itself on the lattice also. In regime 1 the model is well-defined for integer $L \geq 2$. Among others, one specification of regime 1 is the crossing parameter

$$\lambda = \frac{\pi L}{4(L+1)}. \quad (1)$$

The central charge of the dilute A_L model in this regime (or technically, at the critical limit of the regime) is

$$c = 1 - \frac{6}{(L+1)(L+2)}, \quad (2)$$

and the modular invariant partition function is $(A_L, A_{L+1})^{(13, 14)}$ in the classification scheme of ref. 15. In the scaling limit the model realizes the perturbation $\phi_{2,1}$ of the minimal unitary series $\mathcal{M}(L+1, L+2)$.⁽¹²⁾ The elliptic nome p which appears in the face weights of the model corresponds to the coupling constant of the perturbation. For L even the nome is

thermal, and one should distinguish between regime 1^+ ($p > 0$) and regime 1^- ($p < 0$).

The q -state Potts model for $q \leq 4$, on the basis of its critical exponents determined by numerical and renormalization group studies,^(16,17) has also been identified with the minimal unitary series, by way of⁽¹⁸⁾

$$\sqrt{q} = 2 \sin \frac{\pi(t-1)}{2(t+1)} \quad (3)$$

when the central charge is written as

$$c = 1 - \frac{6}{t(t+1)}. \quad (4)$$

The perturbation to which this model corresponds in the scaling limit is again $\phi_{2,1}$. For both models this identification is made from the conformal weight in the leading term of the free energy (see (7) below).

That they are both identified with the same perturbation $\phi_{2,1}$ suggests a relationship between certain members of the dilute A hierarchy and the Potts models as given in Table I, or from inspection of (2) and (4) by naively setting $t = L + 1$. This is not to say that they are the same models, or even that they are in the same universality class. In general the Potts models and the dilute A models have different internal symmetries,⁽¹⁹⁾ different numbers of ground states, and their order parameters may be associated with a different subset of the possible scaling fields of $\mathcal{M}(t, t+1)$. This is the analogue of an idea discussed for three realizations of the $\phi_{1,3}$ perturbation by Delfino.⁽²⁰⁾ Just as there may be more than one S -matrix in the field theory context associated with a particular perturbation, here in the context of statistical mechanics there are two lattice models with only some features in common. For instance, the adjacency diagram of the 3-state Potts model has the symmetry of the D_4 Dynkin diagram, and

Table I. Potts Models and Dilute A_L Models Which Share a Common Central Charge and Critical Exponent α

Central charge	Potts model	Dilute A model	α
$c \rightarrow 1$	$q = 4$	$L \rightarrow \infty$	$2/3$
$c = 4/5$	$q = 3$	$L = 4$	$1/3$
$c = 1/2$	$q = 2$	$L = 2$	0
$c \rightarrow 0$	$q \rightarrow 1$	$L \rightarrow 1$	$-2/3$

the appropriate modular invariant partition function is (A_4, D_4) .^(14, 19) The number of ground states of the dilute A_L model grows with L , but the 4-state Potts model (which we associate with $L \rightarrow \infty$) has four ground states. We should not, then, expect universal quantities for the Potts model which involve one-point functions or susceptibilities to be obtainable via the dilute A model.

However, the universal amplitude ratio associated with the specific heat and the correlation length is⁽¹⁾

$$R_\xi = A^{1/d} \xi_0 \quad (5)$$

where d is the dimension, ξ_0 is the leading term amplitude of the correlation length

$$\xi \simeq \xi_0 \tau^{-\nu},$$

and A comes from the definition of the amplitude of the specific heat

$$C \simeq \frac{A}{\alpha} \tau^{-\alpha}.$$

Expressing A in terms of the leading term coefficient \mathcal{A}_f of the singular part of the free energy,

$$-f_s \simeq \mathcal{A}_f \tau^{2-\alpha},$$

it is possible to re-write (5) as:

$$R_\xi = [\alpha(1-\alpha)(2-\alpha) \mathcal{A}_f]^{1/d} \xi_0. \quad (6)$$

The universality of R_ξ , i.e., its independence of metric factors associated with the reduced temperature $\tau \propto T - T_c$, follows from the scaling relation $2-\alpha = d\nu$. Of course, in what follows for the lattice models we have $d = 2$.

In the language of perturbed conformal field theory, the free energy and the correlation length are related directly to the coupling constant g of the perturbation and the associated conformal weight Δ :

$$f_s \sim g^{d/(d-2\Delta)} \quad \xi \sim g^{-1/(d-2\Delta)}. \quad (7)$$

Thus when attention is confined to the amplitude ratio R_ξ , the required quantities \mathcal{A}_f , ξ_0 and α (or equivalently Δ) relate solely to the perturbing operator. This operator is $\phi_{2,1}$ for both dilute A in regime 1 and the Potts model, and any universal observable associated only to it should be common⁽²⁰⁾ for the points of contact between the models, as shown in Table I.

The singular part of the free energy of the dilute A_L model in regime 1 has been determined using the inversion relation⁽¹²⁾ and exact perturbative^(21, 22) approaches. The leading term is:⁽¹²⁾

$$f_s \sim \begin{cases} p^2 \ln(p) & L = 2 \\ p^{4(L+1)/3L} & L \geq 3 \end{cases}$$

so that for L even

$$\alpha = \frac{2(L-2)}{3L}. \tag{8}$$

Apart from when $L = 2$, the coefficient⁽⁹⁾ is:

$$\mathcal{A}_f = \frac{4\sqrt{3} \sin(2\pi(L-1)/3L)}{\sin(\pi(L-2)/3L)}. \tag{9}$$

The leading term of the correlation length^(21, 22) is

$$\xi^{-1} \simeq 4\sqrt{3} p^{2(L+1)/3L}. \tag{10}$$

Strictly speaking this latter expression was determined for L odd, where it applies both when $p > 0$ and $p < 0$, but there is good reason to believe that it also applies to the high temperature regime for L even. The amplitude ratio found in this paper is thus R_ξ^+ , that is, it applies coming from above the critical temperature.

Substituting the results (8)–(10) into (6), the general expression for this particular universal amplitude ratio of the dilute A_L models in regime 1 is:

$$R_\xi^+ = \left[\frac{2(L-2)(L+1)(L+4)}{27\sqrt{3}L^3} \frac{\sin(2\pi(L-1)/3L)}{\sin(\pi(L-2)/3L)} \right]^{\frac{1}{2}}. \tag{11}$$

Though the discussion above focussed on thermal fields, this expression represents a universal quantity for all L ; for L odd the nome is magnetic-field-like.

Since L , like q , labels the number of states in the model, it would seem it should always be an integer at least equal to 2. However, it has long been realized that q can be treated as a continuous variable, when the Potts model is formulated in terms of the random cluster model.⁽²³⁾ In particular, by taking the limit $q \rightarrow 1$ results for percolation can be obtained. In a similar way we will take $L \rightarrow 1$ in the dilute A model, as foreshadowed in Table I; in the expression for R_ξ^+ (11) there is no impediment to letting L run through all natural numbers. Alternatively one can think of the crossing parameter λ

given in (1) varying quasi-continuously from $\pi/8$ to $\pi/4$. Technical details to do with treating L in this way will be mentioned as necessary, as the four values relevant to the Potts model are now considered.

3. THE UNIVERSAL AMPLITUDE RATIO

3.1. Potts Model with $q = 3, 4$

It is now straight-forward to determine the amplitude ratio between the specific heat, or singular part of the free energy, and the correlation length for the $q = 3$ state Potts model. Setting $L = 4$ in (11):

$$R_{\xi}^{+} = \left[\frac{5}{27\sqrt{3}} \right]^{\frac{1}{2}}. \quad (12)$$

To determine the corresponding amplitude ratio for the $q = 4$ state Potts model, the limit $L \rightarrow \infty$ is taken in (11). The result obtained is

$$R_{\xi}^{+} = \left[\frac{2}{27\sqrt{3}} \right]^{\frac{1}{2}}. \quad (13)$$

3.2. The Ising Model in Zero Magnetic Field, or $q = 2$

It is hardly necessary to obtain an expression for R_{ξ}^{+} for the 2-state Potts model via the dilute A model, since the result is exactly known from the equivalence of this case to the (thermal) Ising model. The results obtained by field theoretic approaches^(2,3) have already been shown to agree with the lattice Ising model values.⁽⁵⁾ In the interests of completeness, then, let us confirm that the known value

$$R_{\xi}^{+} = \frac{1}{\sqrt{2\pi}}$$

for the Ising model in zero magnetic field is recovered from the dilute A_2 model in regime 1.

The expression (9) for the coefficient \mathcal{A}_f of the dilute A_L model does not apply when $L = 2$; in this case correct treatment of the expression for the partition function in refs. 12 or 21 gives

$$f_s \simeq \frac{12}{\pi} p^2 \ln(p).$$

Modifying the definition of the amplitude $C \simeq A \ln(p)$ as is appropriate for the logarithmic divergence, one obtains as expected

$$R_\xi^+ = [2\mathcal{A}_f]^{1/2} \xi_0^+ = \frac{1}{4\sqrt{3}} \left[\frac{24}{\pi} \right]^{1/2} = \frac{1}{\sqrt{2\pi}}.$$

However, the general dilute A_L expression (11) for R_ξ^+ is well-behaved at $L = 2$. Taking the limit $L \rightarrow 2$ gives, correctly,

$$R_\xi^+ = \left[\frac{2(L+1)(L+4)}{9\pi\sqrt{3}L^2} \cos(\pi(L-4)/6L) \right]_{L=2}^{1/2} = \frac{1}{\sqrt{2\pi}}. \tag{14}$$

Incidentally, for this Ising model case, which is related to the minimal unitary conformal field theory $\mathcal{M}(3, 4)$, the scaling field $\phi_{2,1} = \phi_{1,3}$, which can be seen from the identity for conformal weights

$$\Delta_{j,k}^{(4)} = \Delta_{3-j,4-k}^{(4)}.$$

The $(r-1)$ -state models of Andrews, Baxter and Forrester⁽¹⁰⁾ are known⁽²⁴⁾ to realise the $\phi_{1,3}$ perturbation of the minimal unitary series $\mathcal{M}(r-1, r)$ and for $r=4$ should also give the 2-state Potts amplitude ratio under consideration.

The free energy and correlation length of the ABF models, obtained⁽¹⁰⁾ from the 8-vertex model results,⁽²⁵⁾ are:

$$f_s \simeq -4 \cot(\pi^2/2\lambda) \tau^{\pi/2\lambda} \tag{15}$$

$$\xi^{-1} \simeq 4\tau^{\pi/4\lambda}. \tag{16}$$

However, the crossing parameter is $\lambda = \pi/r$ for the ABF models, and for r even the free energy (15) should properly be modified with a logarithmic factor, and the coefficient re-calculated. Instead, simply constructing the amplitude ratio (6) of the coefficients in (15) and (16) and taking $r \rightarrow 4$ by the approach used to obtain (14) for the dilute A model:

$$\begin{aligned} R_\xi^+ &= \lim_{r \rightarrow 4} \left[\frac{(r-4)(r-2)r \cos(\pi r/2)}{32 \sin(\pi r/2)} \right]^{1/2} \\ &= \frac{1}{2} \lim_{r \rightarrow 4} \left[\frac{(r-4)}{\sin(\frac{\pi r}{2} - 2\pi)} \right]^{1/2} = \frac{1}{\sqrt{2\pi}}. \end{aligned}$$

3.3. Percolation, or $q \rightarrow 1$

The percolation result, though previously presented,⁽⁹⁾ is reiterated here for completeness. A review of the relationship between the q -state

Potts model and percolation from the point of view of universal amplitude ratios is given in ref. 3. The appropriate object of interest for percolation is the ratio

$$\tilde{R}_\zeta^+ = \lim_{q \rightarrow 1} \frac{R_\zeta^+}{(q-1)^{1/2}}.$$

To obtain \tilde{R}_ζ^+ from the dilute A model, we put $t = L + 1$ in the expression (3) for q , and then apply trigonometric identities to $(q-1)$:

$$q-1 = 4 \sin\left(\frac{\pi(2L+1)}{3(L+2)}\right) \sin\left(\frac{\pi(L-1)}{3(L+2)}\right).$$

Although there is a factor in the numerator of (11) which becomes zero at $L = 1$, we see that its ratio with $(q-1)$ will be finite in the limit $L \rightarrow 1$, so that

$$\tilde{R}_\zeta^+ = \left[\frac{(L-2)(L+1)(L+4)(L+2)}{27\sqrt{3} L^4 \sin\left(\frac{\pi(2L+1)}{3(L+2)}\right) \sin\left(\frac{\pi(L-2)}{3L}\right)} \right]_{L=1}^{\frac{1}{2}} = \left[\frac{40}{27\sqrt{3}} \right]^{\frac{1}{2}}. \quad (17)$$

4. DISCUSSION

In 1984 Kaufman and Andelman⁽²⁶⁾ presented an argument that the specific heat amplitude ratio (above and below T_c) is $A_+/A_- = 1$ for the q -state Potts model ($q \leq 4$). The free energy expression (9) applies for both signs of p , so that this value of the universal amplitude ratio for the specific heat holds for the dilute A model for all L , including the special cases applicable to the Potts model.

Moreover, an expression was proposed in ref. 26 for the q -dependence of the amplitude of the singular part of the free energy of the Potts model for $q \leq 4$, which accounted for its known divergences and zeroes and which we will denote A_{KA} . Substituting $t = L + 1$ in (3) and this then into A_{KA} it can be shown on rearranging and comparison with (9) that

$$A_{KA} = \frac{b(q)}{6} \mathcal{A}_f.$$

Here $b(q)$ is a positive, slowly-varying function allowed for in ref. 26 so that A_{KA} and \mathcal{A}_f must have common zeroes and divergences. We have already observed, in constructing the amplitude ratio, that \mathcal{A}_f is divergent at $q = 2$ and zero at $q = 1$.

It was remarked below expression (11) that it represented a universal quantity for the dilute A_L model for all L . It is related in a straight-forward manner to the universal quantity considered in quantum field theory ϵm^{-2} , where ϵ is the bulk vacuum energy and the mass m of the field theory is the inverse of the correlation length ξ in the scaling limit of the lattice model. The quantity (see (9) and (10))

$$-f_s \xi^2 = \frac{\sin(2\pi(L-1)/3L)}{4\sqrt{3}\sin(\pi(L-2)/3L)} \tag{18}$$

agrees exactly (when the various notations are translated) with ϵm^{-2} calculated for the $\phi_{2,1}$ perturbed theory by the thermodynamic Bethe ansatz⁽²⁷⁾ and two-kink form factor approach⁽³⁾ based on the S -matrix of Chim and Zamolodchikov.⁽²⁸⁾

Thus the algebraic expressions for R_ξ^+ for the Potts model calculated in this paper from the dilute A model agree precisely with those obtained implicitly by Delfino and Cardy.⁽³⁾ The numerical values given in Tables III and V of ref. 3 (for comparison with previous numerical results for the lattice Potts model) use the ‘‘second moment’’ correlation length, which differs by a few percent from the ‘‘true’’ correlation length used here. This can be seen in Table II where the second moment values⁽³⁾ for R_ξ are reproduced together with evaluations of the exact expressions (12), (13), (14) and (17).

The authors of ref. 3 have further obtained numerical values for other universal amplitude ratios for the Potts models which do not appear to be accessible from solvable lattice models. Their various results are new for $q = 3, 4$ and improve results for percolation ($q \rightarrow 1$) from Monte-Carlo or series enumeration techniques for the lattice Potts model itself. The good

Table II. The Potts Model Universal Amplitude Ratio R_ξ^+ , Determined from Quantum Field Theory in the Two-Kink Approximation to the Form-Factors, Using the Second Moment Correlation Length (by Delfino and Cardy), and from Special Cases of the Dilute A Model Using the True Correlation Length (this Paper)

Potts model	Two-kink approx.	Dilute A model
$q = 4$	0.2052	$2^{1/2}3^{-7/4} = 0.20680\dots$
$q = 3$	0.3262	$5^{1/2}3^{-7/4} = 0.32698\dots$
$q = 2$	0.3989	$2^{-1/2}\pi^{-1/2} = 0.39894\dots$
$q = 1$	0.926	$2^{3/2}5^{1/2}3^{-7/4} = 0.92484\dots$

accuracy of the field theoretic approach was previously discussed in the context of self avoiding walks.⁽²⁹⁾ Nevertheless, it is hoped that this calculation in the solvable model context, though limited to one of the amplitudes, is of interest to field theorists and statistical mechanists alike.

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